

Term Structure of Credit Spreads of A Firm When Its Underlying Assets are Discontinuous

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ABSTRACT

We revisit the previous works of Leland [12], Leland and Toft [11] and Hilberink and Rogers [7] on optimal capital structure and show that the credit spreads of short-maturity corporate bonds can have nonzero values when the underlying of the firm's assets value has downward jumps. We give an analytical treatment of this fact under a general Levy process and discuss some numerical examples under pure jump processes.

Keywords: Optimal capital structure, credit risk, term structure of credit spread

Introduction

The determination of credit spreads has in fact been the ultimate goal of most credit risk models¹. From the structural credit risk models of Black and Scholes [3], Merton [13], Leland [12], and Leland and Toft [11], one can deduce that the credit spreads for short-maturity corporate bonds should always have zero values. Empirical study such as found in Sarig and Warga [15] suggests contrary results that the actual credit spreads do not have such feature. This is due to the fact that the models have relied heavily on diffusion processes for modeling the evolution of the underlying asset of the firm. Quite recently, a class of jump-diffusion processes has been used as an alternative

model. We refer, among others, to Zhou [16], Hilberink and Rogers [7], and Chen and Kou [4]. We consider the following model for a firm based on the earlier works of Leland [12], Leland and Toft [11] and Hilberink and Rogers [7].

To start with, let $V(t)$ denote the value of the firm's asset at time t whose dynamics follow an exponential Levy process². (See Bertoin [1] for further details.)

$$V(t) = Ve^{X_t}.$$

We assume that a default-free asset exists paying a continuous interest rate $r > 0$, and the discounted value $e^{-(r-\delta)t}V(t)$ of

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¹ See Bielecki and Rutkowski [2] for further information.

² The law of a Levy process is specified by its characteristic exponent $\Psi(\lambda) = -\frac{1}{t} \log \mathbb{E}(e^{i\lambda X_t})$, where $i = \sqrt{-1}$, $\Psi(\lambda) = -i\mu\lambda + 1/2\sigma^2\lambda^2 - \int_{-\infty}^{\infty} (e^{i\lambda y} - 1 - i\lambda y 1_{(|y|<1)})\Pi(dy)$, μ and $\sigma \geq 0$ are real constants, and $\int_{-\infty}^{\infty} \min(1, y^2) \Pi(dy) < \infty$.

the firm's asset is \mathbb{P} -martingale under risk-neutral pricing kernel \mathbb{P} ; that is to say that

$$\mathbb{E}[e^{-(r-\delta)t}V(t)] = V, \quad \dots (1)$$

where $\delta > 0$ is the total payout rate to the firm's assets at time t .

The firm is assumed to be partly financed by debt contracted at time zero with face value p maturing at time t and is signed at an agreed fixed contractual interest rate m . We assume that the debt is being constantly retired and reissued. Based on this standing assumption, the total face value of all debt P equals to $P = \int_0^\infty e^{-mt}p dt = p/m$. Furthermore, we assume that all debt is of equal seniority and attracts coupons of an amount ρP , where $\rho > 0$ being the coupon rate, at time t until maturity, or default if that occurs sooner.

The life time of the process $V(t)$ has no time limit unless the value of the firm's asset $V(t)$ falls to a default-triggering level V_B or lower. For the moment we shall assume that V_B is constant. Default happens at the first time $V(t)$ goes below the default level V_B ;

$$\tau_{V_B}^- = \inf\{t > 0: V(t) < V_B\}. \quad \dots (2)$$

Depending on the paths regularity of the underlying process, the default level V_B may be determined endogenously either by using continuous or smooth pasting conditions. We refer to Kyprianou and Surya [10] for further discussion. On default, a fraction η of the value of the firm's asset is also assumed to be lost in reorganizations.

Let us now consider a bond issued at time 0 with face value 1 and maturity t , which continuously pays a constant coupon at a fixed rate $\rho > 0$. Let $1/P$ be the fraction of the asset value $V(\tau_{V_B}^-)$ which debt of maturity t receives in the event of default. The value of the debt with maturity t is therefore given by

$$d(V; V_B, t) = \mathbb{E} \left[\int_0^{\min(t, \tau_{V_B}^-)} \rho e^{-ru} du \right] + \mathbb{E} \left[e^{-rt} 1_{\{t < \tau_{V_B}^-\}} \right] + \frac{1}{P} (1 - \eta) \mathbb{E} \left[e^{-r\tau_{V_B}^-} V(\tau_{V_B}^-) 1_{\{\tau_{V_B}^- \leq t\}} \right].$$

The first term on the right represents the expected discounted value of all coupon payments till time t or the default time $\tau_{V_B}^-$, whichever is sooner. The second term

represents the expected discounted value of the principle repayment, if this occurs before default, and the final term is the net present value of what is recovered upon default, if this happens before maturity time t . Indeed, $V(\tau_{V_B}^-)$ the value of the firm's asset when default occurs and $(1 - \eta)V(\tau_{V_B}^-)$ is the value of the remains after default costs are deducted. Of this, the bondholder with face value 1 gets the fraction $\frac{1}{P}$ since his debt represents this fraction of the total debt outstanding. Notice that if the process X is continuous, then $V(\tau_{V_B}^-)$ would simply be the default level V_B , but since we allow X_t having possibility of jumps, $V(\tau_{V_B}^-)$ can be below the default level V_B .

Term Structure of Credit Spreads

Following Hilberink and Rogers [7], the credit spreads are identified as what coupon would be required to induce an investor to lend one dollar to the firm until maturity time T . This is the interpretation that one would put on a reported credit spread curve for a firm.

By finding the value of coupon rate $\rho = \rho^*$ for which $d(V; V_B, t) = 1$ when $t = T$ and $V(T) = V$, we find the spread $\rho^* - \rho$ for borrowing with fixed maturity T ,

$$CS(T, V; V_B) = r \frac{\mathbb{E} \left[\left(1 - \frac{(1-\eta)}{P} V(\tau_{V_B}^-) \right) e^{-r\tau_{V_B}^-} 1_{\{\tau_{V_B}^- \leq T\}} \right]}{\mathbb{E}[1 - e^{-r\min(T, \tau_{V_B}^-)}]}. \quad \dots (3)$$

The following theorem gives an asymptotic value of the credit spreads $CS(T, V; V_B)$ as the maturity time T decreases to zero. The result, however, was obtained by Hilberink and Rogers [7] for one-sided jump-diffusion processes and was extended to two-sided jump-diffusion processes by Chen and Kou [4]. Note that the jumps structure of the underlying process X considered by these authors is of exponential type.

Adapting the arguments of Hilberink and Rogers [7], we will show below using Proposition 2 on page 7 of Bertoin [1] that the result obtained by the above mentioned authors can in fact be extended to a

general class of stochastic processes, namely Levy processes.

Theorem. Suppose that for a given $\varepsilon > 0$, the Levy measure Π satisfies

$\Pi(-\infty, -\varepsilon) < \infty$. Then, we have

$$\lim_{T \rightarrow 0} CS(T, x) = \bar{\Pi}^-(x) \left[1 - \frac{(1-\eta)}{P} \bar{V}(x) \right].$$

Where we have

defined $x = \log(\frac{V_B}{V})$, $\bar{\Pi}^-(x) =$

$\Pi(-\infty, x)$, and

$\bar{V}(x) = [\bar{\Pi}^-(x)]^{-1} \int_{-\infty}^x V e^y \Pi(dy)$.

Proof To prove the above claim, let us assume that X is a pure jump Levy process, or let us ignore any contribution of the continuous part of a Levy process such as the drift and the Brownian motion to the movements of the Levy process. Thus, the Levy process could have gone below the default level $x = \log(\frac{V_B}{V}) < 0$ only made possible by downward jumps. Furthermore, we denote by $\mathcal{N}(dx, dt)$ the Poisson random measure³ associated with the jumps of the Levy process having compensator $\Pi(dx)dt$. Hence, the random time τ_x^- would be the first entrance time of a jump of the Levy process in the set $(-\infty, x)$, with $x < 0$. It is known that τ_x^- is exponentially distributed with parameter $\bar{\Pi}^-(x)$ (we refer to the Proposition 2 on page 7 in Bertoin [1], and page 143 in Kyprianou [9] for details) since $\mathbb{P}(\tau_x^- > T) = \mathbb{P}(\mathcal{N}(\{-\infty, x\} \times [0, T]) = 0) =$

$e^{-T\bar{\Pi}^-(x)}$. Note that this expression can also be rewritten as $\mathbb{P}(\tau_x^- \leq T) = 1 - e^{-T\bar{\Pi}^-(x)}$. Hence, following the latter equation, we have an approximation as $T \rightarrow 0$ for the probability of default $\mathbb{P}(\tau_x^- \leq T) = T\bar{\Pi}^-(x) + o(T)$. The term $o(T)$ may be thought of as the probability of having more than one jump in the very short period of time $[0, T]$. Given that $\tau_x^- \leq T$, the law of $\log(V(\tau_x^-))$ will be equal to the law of a single jump conditioned to have gone below the level x , and therefore we have that $\mathbb{E}[V(\tau_x^-)|\tau_x^- \leq$

$T] = [\bar{\Pi}^-(x)]^{-1} \int_{-\infty}^x V e^y \Pi(dy)$. It is verifiable to see that $\mathbb{E}[e^{-r\tau_x^-} 1_{\{\tau_x^- \leq T\}}] = \mathbb{P}(\tau_x^- \leq T) + o(T)$, $\mathbb{E}[V(\tau_x^-) e^{-r\tau_x^-} 1_{\{\tau_x^- \leq T\}}] = \mathbb{E}[V(\tau_x^-) | \tau_x^- \leq T] \times \mathbb{P}(\tau_x^- \leq T) + o(T)$ and $\mathbb{E}[1 - e^{-r\min(T, \tau_{V_B}^-)}] = rT + o(T)$ as $T \rightarrow 0$. The proof of our claim follows after taking the limit $\lim_{T \rightarrow 0}$ in the equation (3) above. ■

Numerical Examples

In this section we provide numerical examples of non-zero credit spreads for short-maturity corporate bonds. For this purpose, we assume that the underlying process is generated by downward jumps α -stable processes whose Laplace exponent κ is specified by

$$\kappa(\lambda) = \Psi(i\lambda) = -\frac{1}{t} \log \mathbb{E}[e^{-\lambda X_t}] = K\lambda^\alpha, \alpha \in (1, 2].$$

This type of stochastic processes is a special class of pure jump Levy processes (the case when $\mu = 0$ and $\sigma = 0$) having paths of unbounded variation⁴. We refer to Chapter VII in Bertoin [1] for further details. These processes complement the jump-diffusion processes used recently by Zhou [16], Hilberink and Rogers [7] and Chen and Kou [4]. For numerical study, we choose $\alpha = 1.75$ and $\alpha = 2$. The latter choice of α corresponds to X being a pure Brownian motion.

For all computations, we set the following parameters: $r = 7.5\%$, $\delta = 7\%$, $\eta = 50\%$ and $\tau = 35\%$, which are the values used in [12], [11] and [7]. We also assume as in [11] and [7] that $V_T = \frac{\rho P}{\delta}$. The parameter K in the Laplace exponent $\kappa(\lambda)$ is chosen such that the martingale condition (1) is satisfied.

The plot in Figures FIGURE I and FIGURE II show various shapes of the term structure of credit spreads. Compare the continuous case ($\alpha = 2$, corresponding to X being pure Brownian motion), see FIGURE I, with the pure jump case ($\alpha = 1.75$), see FIGURE II. We notice that the credit spreads go to zero as the time to maturity T tends to zero in the case

³We refer to the Proposition 1.16 in Chapter II of Jacod and

Shiryaev [8] for definition, proofs and further information.

⁴ That is when the Levy measure satisfies $\int_{-1}^1 |y| \Pi(dy) = \infty$.

of pure Brownian motion, but seem to have positive limiting values in the other case. In other respects, the numerical results obtained resemble the similar type of behavior found previously by Sarig and Warga [15], Pitts and Shelby [14], Leland [12], Leland and Toft [11], Hilberink and Rogers [7], and Chen and Kou [4].

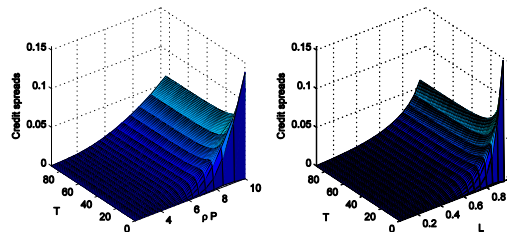


FIGURE I: Shape of credit spreads with debt maturity profile $m = 10$ as a function of maturity T , coupon payment pP , and leverage L running from 5% to 75%. The underlying X is α -stable process with $\alpha = 2$.

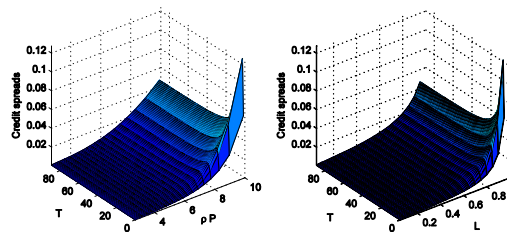


FIGURE II: Shape of credit spreads with debt maturity profile $m = 10$ as a function of maturity T , coupon payment pP , and leverage L running from 5% to 75%. The underlying is α -stable process with $\alpha = 1.75$.

Conclusion

We have built on the previous works of Leland [12], Leland and Toft [11] and Hilberink and Rogers [7] showing that one may push the model considered by these authors fully into the case where the underlying source of randomness of the firm's assets is driven by exponential Levy processes. We show that in the presence of downward jumps, the term structure of credit spreads for short-maturity bonds do not always have zero value. This conclusion agrees with the recent empirical findings of Sarig and Warga [15] and Pitts and Shelby [14] as well as the theoretical findings of Zhou [16], Hilberink and Rogers [7], and Chen and Kou [4] under the model of exponential jump-diffusion processes. We have done this by giving an analytical treatment using the Proposition 2 on page 7 of Bertoin [1]. Moreover, our justification for this discovery goes further than numerical observations and we give a formal proof of this fact.

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